From Witten conjecture to DVV's loop equation

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In this note, I will show that the two conditions (KdV+SE) of the Witten conjecture imply Dijkgraaf-Verlinde-Verlinde's loop equation (see below for details).

Let $Z \in \mathbb{C}[[t_0, t_1, \dots]]$ be the Witten-Kontsevich tau function, and

$$F = \log Z = \sum_{g \geq 0} \hbar^{g-1} F_g$$

be the corresponding free energy. The Witten conjecture [6] states that F is uniquely determined by the following two conditions:

- Let $u = \hbar \, \partial^2 F$, where $\partial = \partial_0$, and $\partial_k = \frac{\partial}{\partial t_k}$. Define a collection of polynomials $R_k(u, u', u'', \dots)$, where the prime stands for the derivative with respect to t_0,

$$R_0 = 1, \quad (k + \frac{1}{2}) R'_{k+1} = u \, R'_k + \frac{1}{2} u' \, R_k + \frac{\hbar}{8} R'''_k, \tag{1}$$

  such that all R_k $(k \geq 1)$ contains no constant terms. Then u satisfies the following Korteweg-de Vries (KdV) hierarchy:

$$\partial_k u = R'_{k+1}.$$

- F satisfies the following String Equation (SE)

$$\partial F = \frac{1}{2\hbar} t_0^2 + \sum_{k \geq 0} t_{k+1} \partial_k F. \tag{2}$$

By definition, we have

$$\begin{aligned} R_1 &= u, \\ R_2 &= \frac{u^2}{2} + \frac{\hbar}{12} u'', \\ R_3 &= \frac{u^3}{6} + \frac{\hbar}{24} (u')^2 + \frac{\hbar}{12} u \, u'' + \frac{\hbar^2}{240} u^{(4)}, \dots \end{aligned}$$

To obtain these polynomials, one must integrate the right hand side of (1) with respect to t_0. An interesting question arises: why is the right hand side always a total derivative of a polynomial of $u, u', u'', \dots$ with respect to t_0?

Lemma 1 *If we define the following generating function of R_k*

$$b(\lambda) = \sum_{k\geq 0} \frac{\Gamma(k+1/2)}{\lambda^{k+1/2}} R_k,$$

then it is determined by the following equation

$$(u-\lambda)b^2 + \frac{\hbar}{8}\left(2\,b\,b'' - (b')^2\right) = -\pi. \tag{3}$$

Proof: The recursion equation of R_k gives

$$(u-\lambda)b' + \frac{u'}{2}b + \frac{\hbar}{8}b''' = 0. \tag{4}$$

Times $2\,b(\lambda)$ on the both sides, and then integrate with respect to t_0, one obtain

$$(u-\lambda)b^2 + \frac{\hbar}{8}\left(2\,b\,b'' - (b')^2\right) = C(\lambda),$$

where $C(\lambda)$ is the generating function of integration constants. Note that all these constants are chozen as zero, so $C(\lambda)$ only has a leading term $-\pi$. The lemma is proved. □

The two conditions of the Witten conjecture are equivalent to the Virasoro constraits for the Witten-Kontsevich tau function [1], that is

$$L_m Z = 0, \quad m \geq -1, \tag{5}$$

where

$$\begin{aligned} L_{-1} &= -\,\partial + \frac{1}{2\hbar}t_0^2 + \sum_{k\geq 0} t_{k+1}\partial_k, \\ L_0 &= -\,\frac{3}{2}\partial_1 + \frac{1}{16} + \sum_{k\geq 0}\left(k+\frac{1}{2}\right)t_k\partial_k, \\ L_m &= -\,\frac{\Gamma(5/2+m)}{\Gamma(3/2)}\partial_{m+1} + \sum_{k\geq 0}\frac{\Gamma(m+k+3/2)}{\Gamma(k+1/2)}t_k\partial_{m+k} \\ &\quad + \frac{\hbar}{2\pi}\sum_{k+l=m-1}\Gamma(k+3/2)\Gamma(l+3/2)\partial_k\partial_l, \quad m\geq 1. \end{aligned}$$

This equivalence is first proved by Dijkgraaf, H. Verlinde, and E. Verlinde in [1], but their original proof lacks some details. Getzler gave a full proof in [2] based on DVV's argument and Virasoro commuting relations. There are also other proofs: Goeree [3] and Kac-Schwartz [4] (based on vertex algebras), or La [5] (based on Lie-Bäcklund transformations). Here I will give another proof which use nothing but the properties of the function $b(\lambda)$. This proof can be regarded as a refinement of DVV's original argument: what we did is just to find out all constants of integration, that are omitted by DVV.

DVV introduced a generating function of all Virasoro constraints, which is called the loop equation for the Witten-Kontsevich tau function. Let $W(\lambda)$ be the following operator

$$W(\lambda) = \sum_{k\geq 0}\frac{\Gamma(k+3/2)}{\lambda^{k+3/2}}\partial_k,$$

which is called the loop operator, and define

$$J(\lambda) = \sum_{k\geq 0} \frac{\lambda^{k-1/2}}{\Gamma(k+1/2)} t_k,$$

and the dilaton shifted one $\tilde{J}(\lambda) = J(\lambda)|_{t_1\to t_1-1}$, then the following generating function of Virasoro constraints

$$\mathcal{L}(\lambda) = \sum_{m\geq -1} \frac{1}{\lambda^{m+2}} \frac{L_m Z}{Z} = 0 \tag{6}$$

can be written as

$$\left(\tilde{J}(\lambda)W(\lambda)(F)\right)_- + \frac{\hbar}{2\pi}\left(W(\lambda)^2(F) + W(\lambda)(F)^2\right) + \frac{t_0^2}{2\,\hbar\,\lambda} + \frac{1}{16\lambda^2} = 0, \tag{7}$$

where $(\quad)_-$ means to take the negative part of a Laurent power series.

The main purpose of this note is to prove (7).

Lemma 2 *Define the following polynomials*

$$B_k(\lambda) = \frac{1}{\Gamma(k+3/2)}\left(\lambda^{k+1/2} b(\lambda)\right)_+,$$

then the k-th KdV equation $\partial_k u = R'_{k+1}$ *is equivalent to the compatibility condition of the following Lax pair:*

$$\phi'' = 2(\lambda - u)\phi,$$
$$\partial_k \phi = \frac{1}{2} B_k \phi' - \frac{1}{4} B'_k \phi.$$

Proof: Check the condition $\partial_k(\phi'') = (\partial_k\phi)''$. □

Lemma 3 *Let* δ *be a derivation that can act on* $b(\lambda)$ *and such that* $[\delta, \partial] = 0$*, then we have the following identities:*

$$\left(1 - \frac{\hbar}{8\pi} b^2\,\partial\, b\,\partial\,\frac{1}{b}\right)\delta(b) = \frac{1}{2\pi}\delta(u-\lambda)b^3, \tag{8}$$

$$\left(1 - \frac{\hbar}{8\pi}\partial\, b\,\partial\, b\right)\delta(\frac{1}{b}) = -\frac{1}{2\pi}\delta(u-\lambda)b. \tag{9}$$

Proof: Act δ on (3), times b, and then use (3) again. □

Lemma 4

i)

$$\partial_k b = \frac{1}{2}\left(B_k\, b' - B'_k\, b\right), \quad \partial_k\left(\frac{1}{b}\right) = \frac{1}{2}\left(\frac{B_k}{b}\right)'.$$

ii)

$$\frac{\partial b}{\partial\lambda} + \frac{\partial b}{\partial u} = 0 \quad \left(\Leftrightarrow \frac{\partial R_{k+1}}{\partial u} = R_k\right).$$

iii)

$$\frac{\delta}{\delta u}\left(\frac{1}{b}\right) = -\frac{1}{2\pi} b$$

iv)

$$\frac{\delta R_{k+1}}{\delta u} = \frac{\partial R_{k+1}}{\partial u} = R_k.$$

Proof: i) Let

$$\partial_k b = \frac{1}{2}\left(B_k\, b' - B_k'\, b\right) + Z,$$

then, by using (8), one can show that

$$\left(1 - \frac{\hbar}{8\pi} b^2\, \partial\, b\, \partial\, \frac{1}{b}\right) Z = 0,$$

so $Z = 0$. ii) is trivial. iii) Choose an arbitrary flow $\partial_t\, u = X$, then (9) implies that

$$\partial_t \left(\frac{1}{b}\right) \equiv -\frac{b}{2\pi} X \quad (\mathrm{mod}\ \mathrm{Im}\, \partial),$$

which is exactly the defining condition of $\frac{\delta}{\delta u}\left(\frac{1}{b}\right)$. iv) The item iii) imples that every R_{k+1} is the variational derivative of another local functional, so we have $\frac{\delta R_{k+1}}{\delta u} = \frac{\partial R_{k+1}}{\partial u}$. The second identity comes from ii). □

Lemma 5 *We have*

$$W(\mu) b(\lambda) = \frac{b(\mu) b(\lambda)' - b(\mu)' b(\lambda)}{2(\mu - \lambda)},$$

$$W(\mu) \left(\frac{1}{b(\lambda)}\right) = \frac{1}{2(\mu - \lambda)} \left(\frac{b(\mu)}{b(\lambda)}\right)'.$$

Proof: $W(\mu) b(\lambda)$ and $W(\mu)\,(1/b(\lambda))$ are just generating functions of $\partial_k b$ and $\partial_k (1/b)$, which have been obtained in Lemma 4 i). □

Lemma 6 *The polynomials R_k's satisfy the following identity:*

$$\partial_k R_{l+1} = \partial_l R_{k+1}.$$

Proof: Because $W(\mu) b(\lambda)$ is symmetric with respect to λ, μ. □

Lemma 7

$$b(\mu)' b(\lambda) = \frac{1}{\mu - \lambda} \left(-\pi\, \frac{b(\mu)}{b(\lambda)} + \frac{\hbar}{8} b(\lambda) \left(b(\lambda) \left(\frac{b(\mu)}{b(\lambda)}\right)'\right)'\right)'$$

Proof: Take $\delta = W(\mu)$, then use Lemma 3. Note that $W(\mu)(u) = b(\mu)'$, the lemma is proved. □

Lemma 8 *$\partial_l R_{k+1}$ is a total derivative of a polynomial of $u, u', u'', \ldots$ with respect to t_0 for all $k, l \geq 0$.*

Proof: Lemma 7 shows that the coefficients of $b(\mu)' b(\lambda)$ are total derivatives, so do the coefficients of $W(\mu) b(\lambda)$ (see Lemma 5). □

We define

$$\Omega_{kl} = \partial^{-1}(\partial_l R_{k+1}),$$

and take the integration constant to be zero. Then construct the following generating function

$$\omega(\mu, \lambda) = \sum_{k,l \geq 0} \frac{\Gamma(k + 3/2)}{\mu^{k+3/2}} \frac{\Gamma(l + 3/2)}{\lambda^{l+3/2}} \Omega_{kl}.$$

Lemma 9

$$\begin{aligned}\omega(\mu,\lambda) =& \frac{\pi}{2(\mu-\lambda)^2}\left(\frac{b(\mu)}{b(\lambda)}+\frac{b(\lambda)}{b(\mu)}-\sqrt{\frac{\lambda}{\mu}}-\sqrt{\frac{\mu}{\lambda}}\right)\\ &-\frac{\hbar}{4\,b(\mu)\,b(\lambda)}\left(\frac{b(\mu)b(\lambda)'-b(\mu)'b(\lambda)}{2(\mu-\lambda)}\right)^2,\\ \omega(\mu,\lambda) =& \frac{\pi}{2(\mu-\lambda)^2}\left(\frac{(\mu-u)+(\lambda-u)}{\pi}b(\mu)b(\lambda)-\sqrt{\frac{\lambda}{\mu}}-\sqrt{\frac{\mu}{\lambda}}\right)\\ &-\frac{\hbar}{8(\mu-\lambda)^2}\left(b(\mu)b(\lambda)''-b(\mu)'b(\lambda)'+b(\mu)''b(\lambda)\right).\end{aligned}$$

Proof: The first identity is just the integration of $W(\mu)b(\lambda)$, since $\omega(\mu,\lambda)' = W(\mu)b(\lambda)$. To compute this integration, we need Lemma 5 and 7. The integration constant is obtained by taking $u = u' = u'' = \cdots = 0$. The second identity is obtained from the first one by using (3). $\square$

Lemma 10

$$\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(b(\mu)) = b(\mu)' + \partial_\mu b(\mu).$$

Proof: Let $\delta = \mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(\cdot)$. According to the string equation, we have

$$\delta(u-\mu) = u' - 1 = (\partial+\partial_\mu)(u-\mu),$$

then Lemma 3 implies $\delta(b(\mu)) = (\partial+\partial_\mu)(b(\mu))$. $\square$

Lemma 11

$$\mathrm{Res}_{\lambda=0}J(\lambda)b(\lambda) = u.$$

Proof:

$$\begin{aligned}&W(\mu)\left(\mathrm{Res}_{\lambda=0}J(\lambda)b(\lambda)\right)\\ =&\mathrm{Res}_{\lambda=0}W(\mu)(J(\lambda))b(\lambda)+\mathrm{Res}_{\lambda=0}J(\lambda)W(\mu)(b(\lambda))\\ =&\mathrm{Res}_{\lambda=0}\left(\sum_{l\geq 0}(l+1/2)\lambda^{l-1/2}\mu^{-l-3/2}\right)b(\lambda)+\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(b(\mu))\\ =&-\partial_\mu b(\mu)+b(\mu)'+\partial_\mu b(\mu)=W(\mu)(u),\end{aligned}$$

so we have $\mathrm{Res}_{\lambda=0}J(\lambda)b(\lambda) = u+C$. Then it is easy to see that $C = 0$ by taking $t_k = 0$. $\square$

Lemma 12 *Suppose $P \in \mathbb{C}[[t_0, t_1, \ldots]]$, if*

$$\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(P) = 0,$$

then P is a constant.

Proof: We learned the idea of the following proof from [2]. Introduce a gradation on $\mathbb{C}[[t_0, t_1, \ldots]]$

$$\deg t_k = k + \frac{1}{2},$$

and rewrite P as a sum of homogeneous components

$$P = \sum_{d\geq 0} P_d, \quad \text{where } \deg P_d = d.$$

Then define

$$\begin{aligned} l_{-1} =& \mathrm{Res}_{\lambda=0} J(\lambda)W(\lambda) = \sum_{k\geq 0} t_{k+1}\partial_k, \\ l_0 =& \sum_{k\geq 0}(k+\frac{1}{2})t_k\partial_k, \\ l_1 =& \sum_{k\geq 0}(k+\frac{1}{2})(k+\frac{3}{2})t_k\partial_{k+1}. \end{aligned}$$

The operators $\{l_{-1}, l_0, l_1\}$ form the basis of an sl_2 Lie algebra, and

$$l_{-1}(P_d) = 0, \quad l_0(P_d) = dP_d, \quad l_1^m(P_d) = 0 \text{ (for } m > d),$$

so P_d gives the highest weight vector of a finite dimentional representation of sl_2. On the other hand, it is easy to see that this representation doesn't contain any negative weight, so it must be the trivial representation, i.e. $l_0(P_d) = 0$. So $P_d = 0$ for any $d > 0$. □

Lemma 13

$$b(\lambda) = \sqrt{\frac{\pi}{\lambda}} + W(\lambda)(\partial F) \quad (\Leftrightarrow R_{k+1} = \hbar\,\partial_k\partial(F)).$$

Proof: Since $\partial_l R_{k+1} = \partial_k R_{l+1}$, there exists a function $G \in \mathbb{C}[[t_0, t_1, \ldots]]$, such that $R_{k+1} = \partial_k G$, so we have

$$b(\lambda) = \sqrt{\frac{\pi}{\lambda}} + W(\lambda)(G).$$

Then by using the string equation and Lemma 11, we obtain

$$\mathrm{Res}_{\lambda=0} J(\lambda)W(\lambda)\,(G - \partial F) = 0,$$

so $G = \partial F + C$ (Lemma 12). □

Lemma 14

$$\omega(\mu,\lambda) = W(\mu)W(\lambda)(F) \quad (\Leftrightarrow \Omega_{kl} = \hbar\,\partial_k\partial_l(F)).$$

Proof: Denote $\delta = \mathrm{Res}_{\lambda=0} J(\lambda)W(\lambda) - \partial_\mu - \partial_\lambda - \partial$, then it is easy to see that

$$\delta(u-\mu) = 0, \quad \delta(u-\lambda) = 0, \quad \delta(b(\mu)) = 0, \quad \delta(b(\lambda)) = 0,$$

so we have

$$\delta(\omega(\mu,\lambda)) = -\frac{\pi}{2(\mu-\lambda)^2}\delta\left(\sqrt{\frac{\lambda}{\mu}} + \sqrt{\frac{\mu}{\lambda}}\right) = -\frac{\pi}{4\mu^{3/2}\lambda^{3/2}}, \tag{10}$$

which is is equivalent to

$$\sum_{i\geq 0} t_{i+1}\partial_i(\Omega_{k,l}) + \Omega_{k-1,l} + \Omega_{k,l-1} = \Omega'_{k,l} - \delta_{k0}\delta_{l0}.$$

In particular, we have

$$(\Omega_{k-1,l} + \Omega_{k,l-1})\,|_{t=0} = \left(\Omega'_{k,l} - \delta_{k0}\delta_{l0}\right)|_{t=0}.$$

On the other hand, by acting $\partial_k\partial_l$ on the string equation, we obtain that

$$\begin{aligned}&\hbar\,(\partial_{k-1}\partial_l(F) + \partial_k\partial_{l-1}(F))\,|_{t=0} = (\hbar\,\partial_k\partial_l\partial(F) - \delta_{k0}\delta_{l0})\,|_{t=0}\\ =&(\partial_k R_{l+1} - \delta_{k0}\delta_{l0})\,|_{t=0} = \left(\Omega'_{k,l} - \delta_{k0}\delta_{l0}\right)|_{t=0},\end{aligned}$$

so we have

$$(\Omega_{k-1,l} + \Omega_{k,l-1})\,|_{t=0} = \hbar\,(\partial_{k-1}\partial_l(F) + \partial_k\partial_{l-1}(F))\,|_{t=0}.$$

Note that $\Omega_{k0} = R_{k+1} = \hbar\,\partial_k\partial(F)$, so we have (by induction)

$$\Omega_{kl}|_{t=0} = \hbar\,\partial_k\partial_l(F)|_{t=0}. \tag{11}$$

The equation (10) also implies that

$$\begin{aligned}\delta(\omega(\mu,\lambda)) = -\frac{\pi}{4\mu^{3/2}\lambda^{3/2}} &= W(\mu)W(\lambda)\left(-\frac{t_0^2}{2}\right)\\ =&\hbar\,W(\mu)W(\lambda)(\delta(F)) = \delta(\hbar\,W(\mu)W(\lambda)(F)).\end{aligned}$$

Here we used the relation $[\delta, W(\mu)W(\lambda)] = 0$, which is not hard to verify. Note that $\omega(\mu,\lambda)' = (\hbar\,W(\mu)W(\lambda)(F))'$, the above identity implies that

$$(\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda) - \partial_\mu - \partial_\lambda)\,(\omega(\mu,\lambda) - \hbar\,W(\mu)W(\lambda)(F)) = 0.$$

Define $Z_{kl} = \Omega_{kl} - \hbar\,\partial_k\partial_l(F)$, the above identity implies that

$$\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(Z_{kl}) = -Z_{k-1,l} - Z_{k,l-1}.$$

We have known that $Z_{00} = 0$. Suppose $Z_{kl} = 0$ for all $k + l < N$, then for k, l satisfying $k + l = N$, we have

$$\mathrm{Res}_{\lambda=0}J(\lambda)W(\lambda)(Z_{kl}) = 0.$$

Lemma 12 implies that Z_{kl} is a constant, then the identity (11) show that $Z_{kl} = 0$. The lemma is proved. □

Remark 15 *According to Lemma 14, the function $\omega(\mu,\lambda)$ is in fact a kind of two-point function. One can define the n-point function $\omega(\lambda_1,\ldots,\lambda_n)$ in the similar way:*

$$\omega(\lambda_1,\ldots,\lambda_n) = \hbar\,W(\lambda_1)\cdots W(\lambda_n)(F).$$

They can be computed by using Lemma 5, 9, and the fact that $W(\lambda)$ is a derivation:

$$\begin{aligned}&\omega(\lambda_1,\ldots,\lambda_n)\\ =&\sum_{i=1}^{n-1}\left(\frac{\partial\omega(\lambda_1,\ldots,\lambda_{n-1})}{\partial b(\lambda_i)}W(\lambda_n)(b(\lambda_i))\right.\\ &\qquad\left.+\frac{\partial\omega(\lambda_1,\ldots,\lambda_{n-1})}{\partial b(\lambda_i)'}W(\lambda_n)(b(\lambda_i)')\right).\end{aligned} \tag{12}$$

Here we used

$$W(\mu)(b(\lambda)') = \frac{b(\mu)b(\lambda)'' - b(\mu)''b(\lambda)}{2(\mu-\lambda)}, \text{ where}$$
$$b(\lambda)'' = \frac{(b(\lambda)')^2}{2b(\lambda)} + \frac{4}{\hbar}\left((\lambda-u)b(\lambda) - \frac{\pi}{b(\lambda)}\right),$$

then one can show that $\omega(\lambda_1,\ldots,\lambda_n)$ $(n \geq 3)$ is always a rational function of λ_i, $b(\lambda_i)$, and $b(\lambda_i)'$ for $i=1,\ldots,n$. For example,

$$\begin{aligned}&\omega(\lambda_1,\lambda_2,\lambda_3)\\ =&\frac{b_1b_1'(b_2^2-b_3^2)+b_2b_2'(b_3^2-b_1^2)+b_3b_3'(b_1^2-b_2^2)}{4(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)b_1b_2b_3}\\ &-\hbar\frac{(b_1b_2'-b_2b_1')(b_2b_3'-b_3b_2')(b_3b_1'-b_1b_3')}{32(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)b_1b_2b_3},\end{aligned}$$

where $b_i = b(\lambda_i)$, $b_i' = b(\lambda_i)'$ for $i=1,2,3$. The initial value $\omega|_{t=0}$ of the n-point function $\omega(\lambda_1,\ldots,\lambda_n)$ also appeared in [7] (up to a certain factor), so we conjecture that there exist certian recurtion relations of Eynard-Orantin's type [7] for the genues expansion of $\omega(\lambda_1,\ldots,\lambda_n)$.

We are ready to prove DVV's loop equation (7). First we prove $L_0(Z)=0$, which gives us a very nice gradation for everything. Then we will use a similar method and this gradation to prove the whole Virasoro constraints, i.e. $\mathcal{L}(\lambda)$=0, where

$$\mathcal{L}(\lambda) = \left(\tilde{J}(\lambda)W(\lambda)(F)\right)_- + \frac{\hbar}{2\pi}\left(W(\lambda)^2(F) + W(\lambda)(F)^2\right) + \frac{t_0^2}{2\,\hbar\,\lambda} + \frac{1}{16\lambda^2}.$$

Recall that $\tilde{J}(\lambda) = J(\lambda)|_{t_1\to t_1-1}$. We also denote $\tilde{t}_k = t_k - \delta_{k1}$.

Lemma 16 *The 0-th Virasoro constraint $L_0(Z)=0$ holds, i.e.*

$$\mathrm{Res}_{\lambda=0}\lambda\tilde{J}(\lambda)W(\lambda)(F) + \frac{1}{16} = 0.$$

Proof: We have known from Lemma 11 that $\mathrm{Res}_{\lambda=0}\tilde{J}(\lambda)b(\lambda)=0$, which implies $\mathrm{Res}_{\lambda=0}\tilde{J}(\lambda)b(\lambda)' = -1$, then by using the identity

$$(\lambda-\mu)W(\lambda)b(\mu) = \frac{1}{2}(b(\lambda)b(\mu)' - b(\lambda)'b(\mu))$$

we obtain

$$\mathrm{Res}_{\lambda=0}\lambda\tilde{J}(\lambda)W(\lambda)(b(\mu)) = \mu\partial_\mu b(\mu) + \frac{1}{2}b(\mu),$$

or equivalently,

$$\sum_{k\geq 0}\left(k+\frac{1}{2}\right)\tilde{t}_k\partial_k(b(\mu)) - \mu\partial_\mu b(\mu) = \frac{1}{2}b(\mu),$$

which means that, if we adopt

$$\deg\tilde{t}_k = k+\frac{1}{2}, \quad \deg\mu = \deg\lambda = -1,$$

then $\deg b(\mu) = \deg b(\lambda) = 1/2$. According to Lemma 9, $\omega(\lambda,\mu)$ has degree 2, so we have

$$\mathrm{Res}_{\lambda=0}\lambda\tilde{J}(\lambda)W(\lambda)(\omega(\mu_1,\mu_2)) = (\mu_1\partial_{\mu_1}+\mu_2\partial_{\mu_2}+2)\,\omega(\mu_1,\mu_2).$$

This equation can be also written as

$$W(\mu_1)W(\mu_2)\Big(\mathrm{Res}_{\lambda=0}\lambda\tilde{J}(\lambda)W(\lambda)(F)\Big) = 0,$$

so there exist constants C, c_k $(k=0,1,2,\dots)$ such that

$$\mathrm{Res}_{\lambda=0}\lambda\tilde{J}(\lambda)W(\lambda)(F) = C + \sum_{k\geq 0} c_k\, t_k.$$

According to the next lemma (Lemma 17), we have

$$\begin{aligned} C &= -\frac{3}{2}\partial_1 F\Big|_{t=0} = -\frac{1}{16}, \\ c_k &= \left(\left(k+\frac{1}{2}\right)\partial_k F - \frac{3}{2}\partial_1\partial_k F\right)\Big|_{t=0} = 0. \end{aligned}$$

The lemma is proved. □

Lemma 17 *For $g\geq 1$, we have*

$$\partial_{3g-2}(\hbar F)|_{t=0} = \frac{1}{g!}\left(\frac{\hbar}{24}\right)^g, \quad \partial_1\partial_{3g-2}(\hbar F)|_{t=0} = \frac{2g-1}{g!}\left(\frac{\hbar}{24}\right)^g,$$

and $\partial_k F|_{t=0} = \partial_1\partial_k F|_{t=0} = 0$ when $3\nmid k+2$.

Proof: These intersection numbers are well-known. Here we give a proof based on the Witten conjecture. According to the string equation, we have

$$\begin{aligned} \partial_k(\hbar F)|_{t=0} &= \partial_{k+1}\partial(\hbar F)|_{t=0} = R_{k+2}|_{t=0}, \\ \partial_1\partial_k(\hbar F)|_{t=0} &= (\partial_1 R_{k+2} - R_{k+2})|_{t=0}, \end{aligned}$$

so we only need to compute $b(\lambda)|_{t=0}$, and $\partial_1 b(\lambda)|_{t=0}$.

Let $\tilde{\beta}(\lambda,x) = b(\lambda)_{t_0=x,t_1=t_2=\cdots=0}$, then $\tilde{\beta}(\lambda,x)$ satisfies

$$(x-\lambda)\tilde{\beta}^2 + \frac{\hbar}{8}(2\,\tilde{\beta}\,\tilde{\beta}_{xx} - \tilde{\beta}_x^2) = -\pi, \quad \tilde{\beta}_x + \tilde{\beta}_\lambda = 0,$$

so we have

$$(x-\lambda)\tilde{\beta}^2 + \frac{\hbar}{8}(2\,\tilde{\beta}\,\tilde{\beta}_{\lambda\lambda} - \tilde{\beta}_\lambda^2) = -\pi.$$

Let $\beta(\lambda) = b(\lambda)|_{t=0} = \tilde{\beta}(\lambda,0)$, then $\beta(\lambda)$ satisfies

$$\lambda\,\beta^2 - \frac{\hbar}{8}(2\,\beta\,\beta_{\lambda\lambda} - \beta_\lambda^2) = \pi. \tag{13}$$

By acting ∂_λ again, we have

$$2\lambda\beta_\lambda + \beta = \frac{\hbar}{4}\beta_{\lambda\lambda\lambda}. \tag{14}$$

From (13) and (14) we can obtain,

$$\beta(\lambda) = \sum_{g\geq 0} \frac{\Gamma(3g+1/2)}{\lambda^{3g+1/2}} \left(\frac{1}{g!}\left(\frac{\hbar}{24}\right)^g\right).$$

According to Lemma 4,

$$\partial_1 b(\lambda) = \frac{1}{3}((2\lambda + u)b(\lambda)' - u'\, b(\lambda)),$$

so we have

$$\begin{aligned}\partial_1 b(\lambda)|_{t=0} =& \frac{1}{3}((2\lambda + u)b(\lambda)' - u'\, b(\lambda))|_{t=0} = \frac{1}{3}((2\lambda\, b(\lambda)' - b(\lambda))|_{t=0} \\ =& -\frac{1}{3}((2\lambda\,\beta_\lambda(\lambda) + \beta(\lambda)) = \sum_{g\geq 0} \frac{\Gamma(3g+1/2)}{\lambda^{3g+1/2}} \left(\frac{2g}{g!}\left(\frac{\hbar}{24}\right)^g\right).\end{aligned}$$

The lemma is proved. □

Lemma 18 *Let $\mathcal{K}(\lambda) = \mathcal{L}(\lambda)'$, then*

$$\mathcal{K}(\lambda) = \left(\tilde{J}(\lambda)b(\lambda)\right)_- + \frac{\hbar}{\pi} W(\lambda)(F)\, b(\lambda) + \frac{\hbar}{2\,\pi} W(\lambda)(b(\lambda)) = 0.$$

Proof: The expression of $\mathcal{K}(\lambda)$ is easy to obtain, so we only prove that it vanishes. Considering $W(\mu)(\mathcal{K}(\lambda))$

$$\begin{aligned}& W(\mu)(\mathcal{K}(\lambda)) \\ =& \left(W(\mu)(\tilde{J}(\lambda))b(\lambda)\right)_- + \left(\tilde{J}(\lambda)W(\mu)(b(\lambda))\right)_- \\ &+ \frac{\hbar}{\pi}\omega(\mu,\lambda)b(\lambda) + \frac{\hbar}{\pi}W(\lambda)(F)\omega(\mu,\lambda)' \\ &+ \frac{\hbar}{2\,\pi}W(\mu)W(\lambda)(b(\lambda)),\end{aligned}$$

one can obtain that

$$\begin{aligned}&(\lambda-\mu)^2\left(W(\mu)(\tilde{J}(\lambda))b(\lambda)\right)_- \\ =& \frac{1}{2}\left(\sqrt{\frac{\lambda}{\mu}} + \sqrt{\frac{\mu}{\lambda}}\right) b(\lambda) - (\lambda-\mu)\partial_\mu b(\mu) - b(\mu),\end{aligned}$$

and

$$\begin{aligned}&(\lambda-\mu)\left(\tilde{J}(\lambda)W(\mu)(b(\lambda))\right)_- \\ =& \partial_\mu b(\mu) + \frac{1}{2}\left(b(\mu)'\mathcal{K}(\lambda) - b(\mu)\mathcal{K}(\lambda)'\right) \\ &+ \frac{\hbar}{2\,\pi}b(\mu)b(\lambda)^2 - \frac{\hbar}{2\,\pi}W(\lambda)(F)(b(\lambda)b(\mu)' - b(\lambda)'b(\mu)) \\ &- \frac{\hbar}{4\,\pi}\left(b(\mu)'W(\lambda)(b(\lambda)) - b(\mu)W(\lambda)(b(\lambda)')\right),\end{aligned}$$

and

$$W(\mu)W(\lambda)(b(\lambda)) = W(\lambda)W(\mu)(b(\lambda)) = W(\lambda)\left(\frac{b(\lambda)b(\mu)' - b(\lambda)'b(\mu)}{2(\lambda-\mu)}\right).$$

Then, after a lenghy computation, one can show that

$$W(\mu)(\mathcal{K}(\lambda)) = \frac{b(\mu)'\mathcal{K}(\lambda) - b(\mu)\mathcal{K}(\lambda)'}{2(\lambda-\mu)}.$$

The left hand side is well-defined when $\mu = \lambda$, so we have

$$b(\lambda)'\mathcal{K}(\lambda) = b(\lambda)\mathcal{K}(\lambda)',$$

then one can show that

$$W(\mu)\left(\frac{\mathcal{K}(\lambda)}{b(\lambda)}\right) = 0,$$

so there exists $C(\lambda)$ such that $\mathcal{K}(\lambda) = C(\lambda)b(\lambda)$. On the other hand (see the proof of Lemma 16), $\deg b(\lambda) = 1/2$, $\deg \mathcal{K}(\lambda) = 3/2$, so $\deg C(\lambda) = 1$, i.e. $C(\lambda) = c/\lambda$. Then it is easy to show that $c = 0$ by checking the leading term of $\mathcal{K}(\lambda)$. □

Theorem 19 *The DVV's loop equation holds true, i.e.* $\mathcal{L}(\lambda) = 0$.

Proof: The idea is very similar to the proof of Lemma 18. We first consider $(\lambda-\mu)^2 W(\mu)(\mathcal{L}(\lambda))$. After a lengthy computation, one can show that $W(\mu)(\mathcal{L}(\lambda)) = 0$, so $\mathcal{L}(\lambda) = C(\lambda)$. Note that $\deg \mathcal{L}(\lambda) = 2$, so $\mathcal{L}(\lambda) = c/\lambda^2$. Then Lemma 16 implies that $c = 0$. The theorem is proved. □

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